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Large time behaviour for functional differential equations with dominant eigenvalues of arbitrary order

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ABSTRACT

The spectral theory for linear autonomous neutral functional differential equations (FDE) yields explicit formulas for the large time behaviour of solutions. Our results are based on resolvent computations and Dunford calculus, applied to establish explicit formulas for the large time behaviour of solutions of FDE. We investigate in detail a class of two-dimensional systems of FDE.

Several aspects of the theory of functional differential equations can be understood as a proper generalization of the theory of ordinary differential equations. However, the fact that the state space for functional differential equations is infinite-dimensional requires the development of methods and techniques from functional analysis and operator theory. The application of the theory of semigroups of operators on a Banach space allows one to use methods from dynamical systems in an infinite-dimensional context. In particular, the perturbation theory, including a variation-of-constants formula, gives rise to a complete theory of invariant manifolds [4,9]. The explicit computation of the flow on the unstable or center manifold requires precise information about the underlying unstable or center subspace of the linearized equation.

This paper is a continuation of the ideas presented in [5]. There it was shown that resolvent computations and Dunford calculus yield explicit formulas for the spectral projection on the unstable or center subspace and, in particular, direct insight in the large time behaviour of both autonomous and periodic functional differential equations. Explicit formulas were given only for simple eigenvalues. In this paper we present the formulas for higher order eigenvalues of the corresponding characteristic equations, and present applications of these formulas in the large time behaviour of autonomous FDE.

We begin introducing functional differential equations and present briefly the corresponding spectral theory. We refer to [5,19] for more details in the spectral theory for these equations. We continue with the computation of the spectral projections using resolvent computations and Dunford calculus. Then we present applications to the large time behaviour of solutions of autonomous FDE, and we study in detail a class of two-dimensional autonomous neutral FDE.

1. Introduction

We denote by $\mathbb{C}^n$ the set of *column* n-vectors with complex-valued entries. We identify the $n \times n$ matrices with complex-valued entries with the linear operators in $\mathbb{C}^n$ and denote this space by $\mathbb{C}^{n\times n}$.

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Let $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}([-r,0],\mathbb{C}^n)$ denote the Banach space of continuous functions from $[-r,0]$ ($r>0$) with values in $\mathbb{C}^n$ endowed with the supremum norm. From the Riesz Representation Theorem (see for instance Rudin [15] or Royden [14]) it follows that every bounded linear mapping $L:\mathcal{C}\to\mathbb{C}^n$ can be represented by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \tag{1.1}$$

where η is a function of bounded variation on $[0,r]$ normalized so that $\eta(0)=0$ and η is continuous from the right in $(0,r)$ with values in the matrix space $\mathbb{C}^{n\times n}$. This set of functions is denoted by $NBV([0,r],\mathbb{C}^{n\times n})$. We can trivially extend $\eta\in NBV([0,r],\cdot)$ in $\mathbb{R}$ by $\eta(\theta)=0$ if $\theta<0$ and $\eta(\theta)=\eta(r)$ if $\theta>r$. In (1.1), the notation $d\eta$ before the integrand φ emphasizes that η is a matrix and φ is a column vector, and therefore the integral is column vector-valued. The variable between parenthesis after the "differential part" is the variable of integration. Sometimes, in order to avoid confusion, we denote the variable of integration as a index of the "d", as in $\int d_\theta[\eta(t+\theta)]f(\theta)$.

As usual in the theory of Delay Equations, for a function x from $[-r,\infty)$ to some Banach space X, we define $x_t:[-r,0]\to X$ by $x_t(\theta)=x(t+\theta)$, $-r\leqslant\theta\leqslant 0$ and $t\geqslant 0$. Derivatives of a function f will be also denoted by Df, where D is viewed as a linear operator. The "dot" notation $\dot f$ will also be used.

1.1. Initial value problem for autonomous FDE

An initial value problem for a linear autonomous Functional Differential Equation (FDE) is given by the following relation

$$\begin{cases} \dfrac{d}{dt}Mx_t = Lx_t, & t\geqslant 0,\\ x_0=\varphi, & \varphi\in\mathcal{C}, \end{cases} \tag{1.2}$$

where $L, M:\mathcal{C}\to\mathbb{C}^n$ are linear continuous, given respectively by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \qquad M\varphi = \varphi(0) - \int_0^r d\mu(\theta)\varphi(-\theta), \tag{1.3}$$

where $\eta,\mu\in NBV([0,r],\mathbb{C}^{n\times n})$ and μ is continuous at zero. See Hale and Verduyn Lunel [9] for a detailed introduction to these equations. As an example, we observe that the differential equation

$$\frac{d}{dt}\big[x(t)+Nx(t-1)\big] = Bx(t)+Cx(t-1), \quad t\geqslant 0,$$

where B, C and N are $n\times n$-matrices, can be written in the form (1.2) by taking $M\varphi=\varphi(0)+N\varphi(-1)$ and $L\varphi = B\varphi(0)+C\varphi(-1)$, what can be done setting $r=1$, $\mu(\theta)=0$ if $\theta<1$, $\mu(\theta)=-N$ for $\theta\geqslant 1$ (*i.e.*, a "jump" of size $-N$ at $\theta=1$), and $\eta(\theta)=0$ for $\theta\leqslant 0$, $\eta(\theta)=B$ for $0<\theta<1$ and $\eta(\theta)=B+C$ for $\theta\geqslant 1$ (the variation of η in (1.2) is concentrated at $\theta=0$ and $\theta=1$, where the "jumps" are, respectively, B and C).

Differential equations of the form (1.2) with $\mu=0$

$$\dot x(t) = Lx_t, \qquad x_0=\varphi$$

are known as *retarded differential equations* or *delay differential equations*. The theory for these equations is well developed (see the books by Hale and Verduyn Lunel [9] and Diekmann et al. [4] for a comprehensive introduction) and form a basis for the theory of neutral equations that we study here.

If for every $\varphi\in\mathcal{C}$ we have existence and uniqueness of continuous solutions of (1.2) for t in some interval $[0,t_0)$, then we can define a semigroup $T(t):\mathcal{C}\to\mathcal{C}$ by

$$T(t)\varphi = x_t, \quad t\in[0,t_0),$$

where x is the solution of (1.2). It is easy to see that T is indeed a strongly continuous semigroup, that is, $T(0)=I$, $T(t+s)=T(t)T(s)$ and $\lim_{t\downarrow 0}T(t)\varphi=\varphi$. The growth bound of the semigroup ensures that the solutions are defined for $t\in[0,\infty)$. The semigroup T is known as the *solution semigroup*.

We take M in the form as in (1.3) with μ continuous at zero as a sufficient condition in order to have well-posedness of the solution semigroup, that is, for the existence and uniqueness of solution through every state. See Hale and Verduyn Lunel [9] for the concept of *atomic at zero*. To derive necessary conditions for the well-posedness is still an open problem. See for instance Ito, Kappel and Turi [10], where the authors present examples of well-posed and non-well-posed linear neutral equations with an M operator which is non-atomic at 0. See also Burns, Herdman and Turi [3].

FDE (1.2) has been studied on a variety of state spaces. O'Connor and Tarn [12,13] studied a class of neutral equations on the Sobolev space $W_2^1([-r,0];\mathbb{R}^n)$ and the controllability and stabilization of solutions. Verduyn Lunel and Yakubovich [20] consider the FDE (1.2) on the space $\mathbb{C}^n\times L^2([-r,0],\mathbb{C}^n)$. Burns, Herdman and Stech [2] and Salamon [16] presented neutral systems with $\mathcal{C}$, $R^n\times L^p$ and $W^{1,p}$ as state spaces.

2. Spectral theory for autonomous FDE

It is convenient to view the FDE (1.2) as an evolutionary system describing the evolution of the state x_t in the Banach space $\mathcal{C}$. In order to do so, we associate with (1.2) a semigroup of solution operators in $\mathcal{C}$. The semigroup is strongly continuous and given by translation along the solution of (1.2)

$$T(t)\varphi = x_t(\,\cdot\,;\varphi),$$

where $x(\,\cdot\,;\varphi)$ denotes the solution of (1.2). See Hale and Verduyn Lunel [9] for further details and more information. The infinitesimal generator A of the semigroup $T(t)$ is given by

$$\begin{cases} \mathcal{D}(A) = \left\{\varphi \in \mathcal{C} \;\Big|\; \dfrac{d\varphi}{d\theta} \in \mathcal{C},\ M\dfrac{d\varphi}{d\theta} = L\varphi\right\}, \\ A\varphi = \dfrac{d\varphi}{d\theta}. \end{cases} \tag{2.1}$$

Let $\lambda \in \sigma(A)$ be an eigenvalue of A. The kernel $\mathcal{N}(\lambda I - A)$ is called the eigenspace at λ and its dimension d_λ, the *geometric multiplicity*. The generalized eigenspace $\mathcal{M}_\lambda$ is the smallest closed subspace that contains all $\mathcal{N}((\lambda I - A)^j)$, $j = 1, 2, \ldots$ and its dimension m_λ is called the *algebraic multiplicity*. It is known that there is a close connection between the spectral properties of the infinitesimal generator A and the characteristic matrix $\Delta(z)$, associated with (1.2), given by

$$\Delta(z) = z\left[I - \int_0^r d\mu(t)e^{-zt}\right] - \int_0^r d\eta(t)e^{-zt}. \tag{2.2}$$

See Diekmann et al. [4] and Kaashoek and Verduyn Lunel [11]. In particular, the geometric multiplicity d_λ equals the dimension of the null space of $\Delta(z)$ at λ and the algebraic multiplicity m_λ is equal to the multiplicity of $z = \lambda$ as a zero of $\det \Delta(z)$. Furthermore, the generalized eigenspace at λ is given by

$$\mathcal{M}_\lambda = \mathcal{N}\big((\lambda I - A)^{k_\lambda}\big), \tag{2.3}$$

where k_λ is the order of $z = \lambda$ as a pole of $\Delta(z)^{-1}$. Using the matrix of cofactors $\operatorname{adj}\Delta(z)$ of $\Delta(z)$, we have the representation

$$\Delta(z)^{-1} = \frac{1}{\det \Delta(z)} \operatorname{adj}\Delta(z). \tag{2.4}$$

From representation (3.1), we immediately derive that the spectrum of A consists of point spectrum only, and is given by the zero set of an entire function

$$\sigma(A) = \big\{\lambda \in \mathbb{C} \;\big|\; \det \Delta(\lambda) = 0\big\}. \tag{2.5}$$

The zero set of the function $\det \Delta(\lambda)$ is contained in a half plane $\{z \mid \operatorname{Re} z < \gamma\}$ in the complex plane. For retarded equations (*i.e.*, $M\varphi = \varphi(0)$), the function $\det \Delta(\lambda)$ has finitely many zeros in strips of the form $S_{\alpha,\beta} = \{z \mid \alpha < \operatorname{Re} z < \beta\}$, where $\alpha, \beta \in \mathbb{R}$. However, in general, for neutral functional differential equations, $\det \Delta(z)$ can have infinitely many zeros in $S_{\alpha,\beta}$. We define a_M as

$$a_M = \inf\big\{\lambda \in \mathbb{R}\colon \sigma(A) \cap S_{\lambda,\infty} \text{ is finite}\big\}. \tag{2.6}$$

An eigenvalue λ of A is called *simple* if $m_\lambda = 1$. So simple eigenvalues of A correspond to the simple roots of the characteristic equation

$$\det \Delta(\lambda) = 0.$$

For $k_\lambda = 1$, in particular if λ is simple, it is known that

$$\mathcal{M}_\lambda = \big\{\theta \mapsto e^{\lambda\theta} v \;\big|\; \theta \in [-r, 0],\ v \in \mathcal{N}\big(\Delta(\lambda)\big)\big\}. \tag{2.7}$$

We refer to Chapter 7 of Hale and Verduyn Lunel [9]. In Kaashoek and Verduyn Lunel [11] and Section IV.3 of Diekmann et al. [4] a systematic procedure has been developed to construct a canonical basis for $\mathcal{M}_\lambda$ using Jordan chains for generic $\lambda \in \sigma(A)$.

From standard spectral theory (see for instance Diekmann et al. [4], Gohberg, Goldberg and Kaashoek [7] and Yosida [22]), it follows that the spectral projection onto $\mathcal{M}_\lambda$ along $\mathcal{R}((\lambda I - A)^{k_\lambda})$ can be represented by a Dunford integral

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (zI - A)^{-1}\, dz, \tag{2.8}$$

where Γ_λ is a small circle such that λ is the only singularity of $(zI - A)^{-1}$ inside Γ_λ. For any $\varphi \in \mathcal{C}$, we have

$$\varphi = P_\lambda \varphi + (I - P_\lambda)\varphi,$$

where $P_\lambda\varphi \in \mathcal{M}_\lambda$. Defining $\mathcal{Q}_\lambda = \mathcal{R}(I - P_\lambda)$, it follows that $\mathcal{C}$ can be decomposed by

$$\mathcal{C} = \mathcal{M}_\lambda \oplus \mathcal{Q}_\lambda.$$

The spaces $\mathcal{M}_\lambda$ and $\mathcal{Q}_\lambda$ are closed subspaces that are invariant under $T(t)$. Likewise, for a set $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues of A, we can write the decomposition

$$\begin{aligned}
&\mathcal{C} = \mathcal{M}_\Lambda \oplus \mathcal{Q}_\Lambda,\\
&\varphi = P_\Lambda\varphi + (I - P_\Lambda)\varphi,\\
&P_\Lambda\varphi \in \mathcal{M}_\Lambda, \qquad (I - P_\Lambda)\varphi \in \mathcal{Q}_\Lambda,
\end{aligned}$$

were $\mathcal{M}_\Lambda = \bigoplus_{i=1}^n \mathcal{M}_{\lambda_i}$, $\mathcal{Q}_\Lambda = \bigcap_{i=1}^n \mathcal{Q}_{\lambda_i}$ and $P_\Lambda\varphi = \sum_{i=1}^n P_{\lambda_i}\varphi$.

Hence the spectral projection can be used to restrict the flow to finite-dimensional subspaces. Banks and Manitius [1], Burns, Herdman and Stech [2] and Verduyn Lunel [17] studied the convergence of the series expansion of the solution operator in terms of the spectral projections into the subspaces $\mathcal{M}_\lambda$.

We finish this section with exponential estimates on the complementary subspace $\mathcal{Q}_{\lambda_d}$ when λ_d is simple and a *dominant eigenvalue of A*, that is, there exists $\epsilon > 0$ such that if λ is another eigenvalue of A, then $\operatorname{Re}\lambda < \operatorname{Re}\lambda_d - \epsilon$.

Lemma 2.1. *Suppose that λ_d is a dominant eigenvalue of A. For $\delta > 0$ sufficiently small there exists a positive constant $K = K(\delta)$ such that*

$$\left\|T(t)(I - P_{\lambda_d})\varphi\right\| \leqslant K e^{(\operatorname{Re}\lambda_d - \delta)t}\|\varphi\|, \quad t \geqslant 0. \tag{2.9}$$

Proof. From the fact that λ_d is dominant, it follows that we can choose $\delta > 0$ sufficiently small such that

$$\sigma(A \mid \mathcal{Q}_{\lambda_d}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < \operatorname{Re}\lambda_d - 2\delta\}.$$

Therefore, the lemma follows from the spectral mapping theorem for retarded functional differential equations (see Theorem IV.2.16 of Diekmann et al. [4]) or from the spectral mapping theorem for neutral equations (see Corollary 9.4.1 of Hale and Verduyn Lunel [9]). $\square$

3. Computing spectral projections

In Section 2 we have presented strong estimates on the complementary space. Later, in Section 4.1, we obtain explicit formulas for the large time behaviour of solutions. Therefore, it is important to be able to compute spectral projections. Our goal is to obtain explicit formulas for the spectral projection for eigenvalues of arbitrary order.

A first approach is to compute spectral projections using duality. In [5] we have done that with help of the bilinear introduced by Hale [8] for delay equations, generalizing it for the context of neutral equations. Although it looks simpler theoretically, it lacks flexibility, because it applies only for simple eigenvalues on autonomous FDE. To compute the spectral projections, another approach is to use Dunford calculus. This technique allows us to provide an explicit formula for the spectral projection on arbitrary order eigenvalues of the generator of the solution semigroup for autonomous FDE, with extensions for a class of periodic equations. It also has the benefit to be able to be applied with FDE other than autonomous. In [5] we have use this method to obtain the large time behaviour of solutions of a class of periodic equations. This approach has the advance of being of algorithmic nature, easing computations when using the so-called computer algebra systems.

In view of (2.8), in order to compute the projection explicitly, we need an explicit formula for the resolvent of A.

Lemma 3.1. *If A is defined by* (2.1)*, then the resolvent $(zI - A)^{-1}$ of A is given by*

$$(zI - A)^{-1}\varphi = -e^{z\cdot}\int_0^{\cdot} e^{-z\tau}\varphi(\tau)\,d\tau + e^{z\cdot}\Delta(z)^{-1}H(z,\varphi), \tag{3.1}$$

where $\Delta(z)$ is given by (2.2) *and $H(z,\cdot): \mathcal{C} \to \mathbb{C}^n$ is an operator given by*

$$H(z,\varphi) = M\varphi + \int_0^r d_\theta\big[z\mu(\theta) + \eta(\theta)\big]\int_0^\theta e^{-z\tau}\varphi(\tau - \theta)\,d\tau. \tag{3.2}$$

Furthermore, if $\lambda \in \sigma(A)$, then the spectral projection P_λ onto $\mathcal{M}_\lambda$ along $\mathcal{R}((\lambda I - A)^{k_\lambda})$ is given by

$$P_\lambda\varphi = \operatorname*{Res}_{z=\lambda}\big\{e^{z\cdot}\Delta(z)^{-1}H(z,\varphi)\big\}. \tag{3.3}$$

Proof. Let φ be fixed. If we define $\psi = (zI - A)^{-1}\varphi$, then $\psi \in \mathcal{D}(A)$ and $z\psi - A\psi = \varphi$. From the definition of A, it follows that ψ satisfies the ordinary differential equation

$$z\psi - \frac{d\psi}{d\theta} = \varphi \tag{3.4}$$

with boundary condition

$$M\frac{d\psi}{d\theta} = L\psi. \tag{3.5}$$

Eq. (3.4) yields

$$\psi(\theta) = e^{z\theta}\left[\psi(0) - \int_0^\theta e^{-z\tau}\varphi(\tau)\,d\tau\right]. \tag{3.6}$$

Applying M on both sides of (3.4) and using (3.5), we obtain

$$\begin{aligned}
0 &= zM\psi - L\psi - M\varphi \\
&= z\left[\psi(0) - \int_0^r d\mu(\theta)\psi(-\theta)\right] - \int_0^r d\eta(\theta)\psi(-\theta) - M\varphi \\
&= z\psi(0) - z\int_0^r d\mu(\theta)\left(e^{-z\theta}\left[\psi(0) - \int_0^{-\theta} e^{-z\tau}\varphi(\tau)\,d\tau\right]\right) - \int_0^r d\eta(\theta)\left(e^{-z\theta}\left[\psi(0) - \int_0^{-\theta} e^{-z\tau}\varphi(\tau)\,d\tau\right]\right) - M\varphi \\
&= \left[zI - z\int_0^r d\mu(\theta)e^{-z\theta} - \int_0^r d\eta(\theta)e^{-z\theta}\right]\psi(0) - M\varphi - \int_0^r d_\theta\big[z\mu(\theta) + \eta(\theta)\big]\int_0^\theta e^{-z\tau}\varphi(\tau - \theta)\,d\tau \\
&= \Delta(z)\psi(0) - H(z,\varphi),
\end{aligned}$$

where $H(z,\varphi)$ is given by formula (3.2). This allows us to solve for $\psi(0)$ and

$$\psi(0) = \Delta(z)^{-1}H(z,\varphi). \tag{3.7}$$

Substituting (3.7) into (3.6) yields (3.1). The formula for the projection in (2.8) is precisely the residue of the resolvent of A in $z = \lambda$, and since

$$z \mapsto e^{z\cdot}\int_\cdot^0 e^{-z\tau}\varphi(\tau)\,d\tau$$

is analytic, the representation for the resolvent in (3.1) yields formula (3.3). $\square$

From the representation (3.1) of the resolvent of A it becomes evident the representation (2.5) for the spectrum $\sigma(A)$.
For the case where λ is a simple eigenvalue of A, we arrive in a simple manner to the formula of the spectral projection.

Theorem 3.2. *Let A be given by* (2.1)*. If λ is a simple eigenvalue of A, then the spectral projection P_λ onto $\mathcal{M}_\lambda(A)$ along $\mathcal{R}((\lambda I - A)^{k_\lambda})$ can be written explicitly as follows*

$$P_\lambda\varphi = e^{\lambda\cdot}\left[\frac{d}{dz}\det\Delta(\lambda)\right]^{-1}\operatorname{adj}\Delta(\lambda)H(\lambda,\varphi), \tag{3.8}$$

where $\operatorname{adj}\Delta(\lambda)$ *denotes the matrix of cofactors of* $\Delta(\lambda)$ *and* $H(z,\varphi)$ *is given by* (3.2)*.*

Proof. If λ is a simple eigenvalue of A, then λ is a simple zero of $\det\Delta(z)$. In order to compute the residue on (3.3), it suffices to compute $\operatorname{Res}_{z=\lambda}\Delta(z)^{-1}$ explicitly

$$\begin{aligned}
\operatorname*{Res}_{z=\lambda}\Delta(z)^{-1} &= \lim_{z\to\lambda}(z-\lambda)\big[\det\Delta(z)\big]^{-1}\operatorname{adj}\Delta(z) \\
&= \lim_{z\to\lambda}\left[\frac{\det\Delta(z) - \det\Delta(\lambda)}{z-\lambda}\right]^{-1}\operatorname{adj}\Delta(z) \\
&= \left[\frac{d}{dz}\det\Delta(\lambda)\right]^{-1}\operatorname{adj}\Delta(\lambda)
\end{aligned}$$

in order to arrive at (3.8). $\square$

In order to continue our computations for eigenvalues of higher order, we provide a formula for a pole of $\Delta(z)^{-1}$ of arbitrary order. We begin defining scalar functions $\phi_{\lambda,n}$, for $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_+$, by

$$t \mapsto \phi_{\lambda,n}(t) = \frac{t^n}{n!} e^{\lambda t}$$

and we state some auxiliary lemmas.

Lemma 3.3. *The functions $\phi_{\lambda,n}$, $n \in \mathbb{Z}_+$ satisfy the following properties*:

(a) (*Derivative*)

$$D\phi_{\lambda,0}(t) = \lambda\phi_{\lambda,0}(t),$$
$$D\phi_{\lambda,n}(t) = \lambda\phi_{\lambda,n}(t) + \phi_{\lambda,n-1}(t), \quad n \geqslant 1.$$

(b) (*Primitive*) *For $\lambda \neq 0$,*

$$\int \phi_{\lambda,n}(t)\,dt = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{\lambda^{n+1-j}} \phi_{\lambda,j}(t) + k, \quad k \text{ constant.}$$

(c) (*Evaluation of sums*)

$$\phi_{\lambda,n}(t_0 + \theta) = \sum_{j=0}^{n} \phi_{\lambda,n-j}(t_0)\phi_{\lambda,j}(\theta).$$

(d) (*Linear operators*) *For $v \in \mathbb{C}^n$ and $t \in \mathbb{R}$, we have that $(\phi_{\lambda,n})_t v \in \mathcal{C}$. Furthermore, for any linear operator B on $\mathcal{C}$ into any complex linear space, we have*

$$B\big[(\phi_{\lambda,n})_t v\big] = \sum_{j=0}^{n} \phi_{\lambda,n-j}(t) B\big[(\phi_{\lambda,j})_0 v\big].$$

Proof. Item (a) is clear. Item (b) follows from integration by parts. For item (c), note that

$$\begin{aligned}\phi_{\lambda,n}(t+\theta) &= \frac{1}{n!}(t+\theta)^n e^{\lambda(t+\theta)} = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} t^{n-j}\theta^j e^{\lambda t} e^{\lambda\theta} \\ &= \sum_{j=0}^{n} \frac{t^{n-j}}{(n-j)!} e^{\lambda t} \frac{\theta^j}{j!} e^{\lambda\theta} = \sum_{j=0}^{n} \phi_{\lambda,n-j}(t)\phi_{\lambda,j}(\theta).\end{aligned}$$

Item (d) follows from item (c) and from the fact that $\phi_{\lambda,n}$ are scalar functions. □

Next we characterize the residue of the product of $\Delta(z)^{-1}$ by an arbitrary Fréchet differentiable mapping, at a pole of $\Delta(z)^{-1}$ of arbitrary order.

Lemma 3.4. *Let $E : \mathbb{C} \to \mathcal{C}$ be mapping which is Fréchet differentiable on $z = \lambda$. If λ is a pole of $\Delta^{-1}(z)$ order n, that is, if*

$$\Delta(z)^{-1} = \frac{K(z)}{(z-\lambda)^n}$$

with $K(z)$ differentiable on a neighbourhood of $z = \lambda$ and $K(\lambda) \neq 0$, then

$$\operatorname*{Res}_{z=\lambda} \Delta(z)^{-1} E(z) = \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!} \frac{D^j E(\lambda)}{j!}. \tag{3.9}$$

Proof. We write down the Laurent expansions of $K(z)$, $\Delta(z)^{-1}$ and $E(z)$

$$K(z) = K_0 + K_1(z-\lambda) + K_2(z-\lambda)^2 + \cdots,$$
$$\Delta^{-1}(z) = (z-\lambda)^{-n}\big(K_0 + K_1(z-\lambda) + K_2(z-\lambda)^2 + \cdots\big),$$
$$E(z) = E_0 + E_1(z-\lambda) + E_2(z-\lambda)^2 + \cdots,$$

where $K_i = D^i K(\lambda)/i!$ and $E_i = D^i E(\lambda)/i!$, $i = 0, 1, 2, \ldots$. Then the Laurent expansion for $\Delta^{-1}(z)E(z)$ is given by

$$\Delta^{-1}(z)E(z) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} K_{k-j}E_j\right)(z-\lambda)^{k-n}$$

and we obtain that the coefficient of the term $(z-\lambda)^{-1}$ is given by

$$\sum_{j=0}^{n-1} K_{n-1-j}E_j = \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\frac{D^j E(\lambda)}{j!}.$$

This yields the representation of the residue of $\Delta^{-1}(z)E(z)$ at $z = \lambda$, given in (3.9). □

Lemma 3.5. *Suppose that a_i, b_i and c_i, $i = 0, \ldots, n$ are elements of a ring and that a_i commutes with b_j for $i, j = 0, \ldots, n$. Then*

$$\sum_{i=0}^{n}\left\{a_i \sum_{j=0}^{i} b_{i-j}c_j\right\} = \sum_{j=0}^{n}\left\{b_j \sum_{i=j}^{n} a_i c_{i-j}\right\}. \tag{3.10}$$

Proof. We apply the change of variables $j \mapsto k = i - j$ in steps (3.11) and an interchanging of order of summation in step (3.12).

$$\begin{aligned}
\sum_{i=0}^{n}\left\{a_i \sum_{j=0}^{i} b_{i-j}c_j\right\} &= \sum_{i=0}^{n}\sum_{j=0}^{i} a_i b_{i-j} c_j = \sum_{i=0}^{n}\sum_{j=0}^{i} b_{i-j} a_i c_j \\
&= \sum_{i=0}^{n}\sum_{k=0}^{i} b_k a_i c_{i-k} \qquad (3.11)\\
&= \sum_{k=0}^{n}\sum_{i=k}^{n} b_k a_i c_{i-k} = \sum_{k=0}^{n}\left\{b_k \sum_{i=k}^{n} a_i c_{i-k}\right\}. \qquad (3.12)
\end{aligned}$$

This shows (3.10). □

Theorem 3.6. *Suppose that λ is a pole of $\Delta(z)^{-1}$ of order n, that is, there exists a regular function K such that $K(\lambda) \neq 0$ and $\Delta(z)^{-1}$ is given by*

$$\Delta(z)^{-1} = \frac{K(z)}{(z-\lambda)^n}. \tag{3.13}$$

The spectral projection P_λ of $\mathcal{C}$ into $\mathcal{M}_\lambda$ along $\mathcal{R}((\lambda I - A)^{k_\lambda})$ is given by

$$P_\lambda \varphi = \sum_{j=0}^{n-1} q_j\, \theta^j e^{\lambda\theta}, \tag{3.14}$$

where $q_j = q_j(n, \lambda, \varphi)$ is given by

$$q_j(n, \lambda, \varphi) = \frac{1}{j!}\sum_{k=j}^{n-1} \frac{D^{n-1-k}K(\lambda)}{(n-1-k)!}\frac{D_1^{k-j}H(\lambda, \varphi)}{(k-j)!}. \tag{3.15}$$

Proof. From Eq. (3.3) we have that

$$P_\lambda \varphi = \operatorname*{Res}_{z=\lambda}\left\{\Delta(z)^{-1}\left[e^{z\cdot}H(z, \varphi)\right]\right\}. \tag{3.16}$$

For the form of Δ^{-1} from the hypothesis, we can use Lemma 3.4 in order to compute the resolvent in Eq. (3.16). We get that

$$(P_\lambda \varphi)(\theta) = \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\frac{1}{j!}\frac{d^j}{d\lambda^j}\left[e^{\lambda\theta}H(\lambda, \varphi)\right]. \tag{3.17}$$

We can further simplify Eq. (3.17) by using Leibniz Formula, that is, the nth derivative of the product of functions $f(t)$ and $g(t)$ is given by

$$\frac{d^n}{dt^n}\big[f(t)g(t)\big] = \sum_{k=0}^{n} \binom{n}{k} D^k f(t) D^{n-k} g(t),$$

and Lemma 3.5 in order to obtain

$$\begin{aligned}
(P_\lambda\varphi)(\theta) &= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\frac{1}{j!}\frac{d^j}{d\lambda^j}\big[e^{\lambda\theta}H(\lambda,\varphi)\big] \\
&= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\frac{1}{j!}\sum_{k=0}^{j}\binom{j}{k}\frac{d^{j-k}}{d\lambda^{j-k}}e^{\lambda\theta}\frac{d^k}{d\lambda^k}H(\lambda,\varphi) \\
&= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\sum_{k=0}^{j}\frac{\theta^{j-k}}{(j-k)!}e^{\lambda\theta}\frac{D_1^k H(\lambda,\varphi)}{k!} \\
&= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\sum_{k=0}^{j}\phi_{\lambda,j-k}(\theta)\frac{D_1^k H(\lambda,\varphi)}{k!} \\
&= \sum_{j=0}^{n-1}\phi_{\lambda,j}(\theta)\sum_{k=j}^{n-1}\frac{D^{n-1-k}K(\lambda)}{(n-1-k)!}\frac{D_1^{k-j}H(\lambda,\varphi)}{(k-j)!}.
\end{aligned}$$

This shows (3.14). □

In order to compute the spectral projection for eigenvalues of second or higher orders, we also need several derivatives of $H(z,\varphi)$, given in formula (3.2), with respect to the first variable.

Lemma 3.7. *For integer $n \geqslant 1$, the nth Fréchet derivative of $H(\lambda,\varphi)$ with respect to the first variable is given by*

$$\begin{aligned}
D_1^n H(z,\varphi) &= (-1)^{n+1} n \int_0^r d\mu(\theta)\int_0^\theta \tau^{n-1}e^{-z\tau}\varphi(\tau-\theta)\,d\tau + (-1)^n z\int_0^r d\mu(\theta)\int_0^\theta \tau^n e^{-z\tau}\varphi(\tau-\theta)\,d\tau \\
&\quad + (-1)^n\int_0^r d\eta(\theta)\int_0^\theta \tau^n e^{-z\tau}\varphi(\tau-\theta)\,d\tau. \qquad (3.18)
\end{aligned}$$

Proof. Let $\Lambda_n(z,\varphi)$ be the expression in the right-hand side of (3.18). We differentiate (3.2) with respect to z and we obtain

$$\begin{aligned}
D_1 H(z,\varphi) &= \int_0^r d\mu(\theta)\int_0^\theta e^{-z\tau}\varphi(\tau-\theta)\,d\tau - z\int_0^r d\mu(\theta)\int_0^\theta \tau e^{-z\tau}\varphi(\tau-\theta)\,d\tau - \int_0^r d\eta(\theta)\int_0^\theta \tau e^{-z\tau}\varphi(\tau-\theta)\,d\tau \\
&= \Lambda_1(z,\varphi).
\end{aligned}$$

It is easy to check that for $n \geqslant 1$, we have

$$\frac{d}{dz}\Lambda_n(z,\varphi) = \Lambda_{n+1}(z,\varphi).$$

By induction, we have that (3.18) holds. □

Example 3.8. Consider the retarded equation

$$\dot{x}(t) = Bx(t-1), \quad t \geqslant 0, \qquad x_0 = \varphi \in \mathcal{C}, \qquad (3.19)$$

where $B \neq 0$ is an $n \times n$-matrix. The characteristic equation is given by

$$\Delta(z) = zI - Be^{-z}. \qquad (3.20)$$

For every simple root of $\det \Delta(z)$, the spectral projection is given by

$$(P_\lambda \varphi)(\theta) = \left[\frac{d}{dz}\det \Delta(\lambda)\right]^{-1} \operatorname{adj} \Delta(\lambda)\left(\varphi(0) + B\int_0^1 e^{-\lambda\tau}\varphi(\tau-1)\,d\tau\right)e^{\lambda\theta}.$$

In the scalar case, a root λ of Δ is not simple if and only if

$$\begin{cases} \lambda - Be^{-\lambda} = 0, \\ 1 + Be^{-\lambda} = 0. \end{cases}$$

Therefore, if $B \neq -1/e$ or equivalently $\lambda = -1$ is not a root of Δ, then all roots of (3.20) are simple. So the spectral projections are given by

$$(P_\lambda \varphi)(\theta) = \frac{1}{1+\lambda}\left(\varphi(0) + B\int_0^1 e^{-\lambda\tau}\varphi(\tau-1)\,d\tau\right)e^{\lambda\theta}, \tag{3.21}$$

where λ satisfies $\lambda - Be^{-\lambda} = 0$. Furthermore, it follows from Corollary 3.12 of Verduyn Lunel [18] that

$$x_t(\varphi) = \sum_{j=0}^{\infty} P_{\lambda_j} T(t)\varphi = \sum_{j=0}^{\infty} T(t) P_{\lambda_j}\varphi, \quad t > 0,$$

where λ_j, $j = 0, 1, \ldots,$ denote the roots of $\lambda - Be^{-\lambda} = 0$, ordered according to decreasing real part. Using (3.21) and the fact that $T(t)e^{\lambda_j \cdot} = e^{\lambda_j(t+\cdot)}$, we can now explicitly compute the solution of (3.19) with initial condition $x_0 = \varphi$

$$x(t;\varphi) = \sum_{j=0}^{\infty} \frac{1}{1+\lambda_j}\left(\varphi(0) + B\int_0^1 e^{-\lambda_j\tau}\varphi(\tau-1)\,d\tau\right)e^{\lambda_j t}, \quad t > 0.$$

(Compare Theorem 6 in Wright [21].)

If $B = -1/e$, then all zeros of $\Delta(z)$ are simple except for $\lambda = -1$. For the simple zeros we can again use (3.21). For the double zero $\lambda = -1$, we have to use (3.3) to compute the projection onto the two-dimensional space $\mathcal{M}_{-1}$ and P_{-1} is given by

$$(P_{-1}\varphi)(\theta) = \frac{2}{3}\left(\varphi(0) + \int_0^1 (3\tau-1)e^{\tau-1}\varphi(\tau-1)\,d\tau\right)e^{-\theta} + 2\left(\varphi(0) - \int_0^1 e^{\tau-1}\varphi(\tau-1)\,d\tau\right)\theta e^{-\theta}.$$

Since $T(t)\phi = \phi(t+\cdot)$, where $\phi(\theta) = \theta e^{-\theta}$, we can again give the solution explicitly

$$\begin{aligned} x(t;\varphi) = {} & \frac{2}{3}\left(\varphi(0) + \int_0^1 (3\tau-1)e^{\tau-1}\varphi(\tau-1)\,d\tau\right)e^{-t} + 2\left(\varphi(0) - \int_0^1 e^{\tau-1}\varphi(\tau-1)\,d\tau\right)te^{-t} \\ & + \sum_{j=1}^{\infty}\frac{1}{1+\lambda_j}\left(\varphi(0) - \int_0^1 e^{-\lambda_j\tau-1}\varphi(\tau-1)\,d\tau\right)e^{\lambda_j t}, \quad t > 0, \end{aligned}$$

where λ_j, $j = 1, 2, \ldots$ are the simple zeros of Δ ordered according to decreasing real part.

4. Applications to the large time behaviour

In order to make the analysis of the characteristic equations easier, we assume throughout the rest of the paper that the coefficients of the functional differential equations are real-valued.

4.1. Autonomous FDE

In this section we shall further investigate the case when $\det \Delta(z)$ has a dominant root $z = \lambda_d$.

Theorem 4.1. *Consider the FDE* (1.2)–(1.3) *and let* $\Delta(z)$ *be as in* (2.2)*. Suppose that* $\det \Delta(z)$ *has a dominant simple zero* λ_d, i.e., *suppose that* $\det \Delta(\lambda_d) = 0$ *and there exists* $\delta > 0$ *such that if* $\det \Delta(\lambda) = 0$ *and* $\lambda \neq \lambda_d$, *then* $\operatorname{Re}\lambda < \operatorname{Re}\lambda_d - \delta$*. Then there exist positive numbers* ϵ *and* N *such that*

$$\left\| e^{-\lambda_d t} T(t)\varphi - P_{\lambda_d}\varphi \right\| \leqslant N e^{-\epsilon t} \tag{4.1}$$

and

$$\lim_{t\to\infty} e^{-\lambda_d t}T(t)\varphi = e^{\lambda_d \cdot}\left[\frac{d}{dz}\det\Delta(\lambda_d)\right]^{-1}\operatorname{adj}\Delta(\lambda_d)\left[M\varphi + \int_{-r}^{0}\left[\lambda_d\,d\mu(\tau)+d\eta(\tau)\right]\int_{0}^{-\tau} e^{-\lambda_d\sigma}\varphi(\sigma+\tau)\,d\sigma\right]. \tag{4.2}$$

Proof. From representation of the spectral projection P_λ on (2.8), it follows that P_λ and A commute, and therefore P_λ and $T(t)$ commute as well. The spectral decomposition with respect to λ_d yields

$$e^{-\lambda_d t}T(t)\varphi = e^{-\lambda_d t}T(t)P_{\lambda_d}\varphi + e^{-\lambda_d t}T(t)(I-P_{\lambda_d})\varphi.$$

From the exponential Lemma 2.1, it follows that there exist positive ϵ and N such that

$$\left\|e^{-\lambda_d t}T(t)(I-P_{\lambda_d})\varphi\right\| \leqslant Ne^{-\epsilon t}, \quad t \geqslant 0.$$

The action of $T(t)$ restricted to a one-dimensional eigenspace $\mathcal{M}_{\lambda_d}$ is given by

$$e^{-\lambda_d t}T(t)P_{\lambda_d} = P_{\lambda_d}.$$

This shows (4.1) and using Lemma 3.2, we arrive at (4.2). □

If we evaluate (4.2) at $\theta = 0$ we obtain the following corollary.

Corollary 4.2. *Consider the FDE* (1.2)–(1.3) *and let* $\Delta(z)$ *be as in* (2.2) *and let* λ_d *to be a simple dominant zero of* $\det\Delta(z)$. *If* $x(t) = x(t;\varphi)$ *denotes the solution of* (1.2) *with initial data* $x_0 = \varphi$, *then the large time as a function of the initial data* φ *is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \left[\frac{d}{dz}\det\Delta(\lambda_d)\right]^{-1}\operatorname{adj}\Delta(\lambda_d)\left[M\varphi + \int_{-r}^{0}\left[\lambda_d\,d\mu(\tau)+d\eta(\tau)\right]\int_{0}^{-\tau} e^{-\lambda_d\sigma}\varphi(\sigma+\tau)\,d\sigma\right]. \tag{4.3}$$

The next theorem, that gives a result similar to the Corollary 4.2, but can be applied to real dominant eigenvalues that are not simple.

Theorem 4.3. *Consider the FDE* (1.2)–(1.3) *and let* $\Delta(z)$ *be as in* (2.2) *and let* λ_d *to be a real dominant zero of* $\det\Delta(z)$ *of geometric multiplicity* $n \geqslant 1$. *If* $x(t) = x(t;\varphi)$ *denotes the solution of* (1.2) *with initial data* $x_0 = \varphi$, *then the large time behaviour as a function of the initial data* φ *is described as follows.*

1. *If* $P_{\lambda_d}\varphi \neq 0$, *then*

$$\lim_{t\to\infty}\frac{1}{t^m}e^{-\lambda_d t}x(t) = q_m(n,\lambda_d,\varphi), \tag{4.4}$$

where $m = \max\{j \in \{0,1,\dots,n-1\}\colon\ q_j(n,\lambda_d,\varphi) \neq 0\}$, *with* q_j *given by formula* (3.15) *in Theorem* 3.6.

2. *If* $P_{\lambda_d}\varphi = 0$, *then*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = 0. \tag{4.5}$$

Proof. We can use the same arguments as in the proof of Theorem 4.1, but now the action of $T(t)$ on the nth dimensional generalized eigenspace $\mathcal{M}_{\lambda_d}$ is given by

$$e^{-\lambda_d t}\big(T(t)P_{\lambda_d}\varphi\big)(0) = \sum_{j=0}^{n-1} q_j t^j, \tag{4.6}$$

where the "exponential times polynomial" form of P_{λ_d} is given by Theorem 3.6. Analyzing the large time behaviour of polynomial on (4.6) yields relations (4.4) and (4.5). □

Example 4.4. We return to Example 3.8, in the scalar case of the delay equation (3.19), and study the large time behaviour of its solutions.

If $B > -1/e$, then by applying Lemma B.3 of [5] to (3.20), we obtain that the unique real root λ of (3.20) is dominant. From Corollary 4.2, it follows that the large time behaviour of solutions of (3.19) is given

$$\lim_{t\to\infty} e^{-\lambda t}x(t) = \frac{1}{1+\lambda}\left[\varphi(0) + \lambda\int_{-1}^{0} e^{-\lambda s}\varphi(s)\,ds\right]. \tag{4.7}$$

Now consider the case $B = -1/e$. Then $\lambda = -1$ is a root of (3.20). We shall show that $\lambda = -1$ is a dominant eigenvalue, that is, $\operatorname{Re}\lambda_j < -1 + \epsilon$ for some $\epsilon > 0$. For $-1/e < \tilde{B} < 0$, it is easy to see that

$$\lambda - \tilde{B}e^{-\lambda} = 0 \tag{4.8}$$

has exactly two negative roots, and both tend to -1 as $\tilde{B} \downarrow -1/e$. Applying Lemma B.3 of [5] to (4.8) with $-1/e < \tilde{B} < 0$, we get that Eq. (4.8) has a real dominant eigenvalue λ_d. By continuity of roots of (4.8) with respect to $\tilde{B}$, we conclude that all roots of (4.8), with $\tilde{B} = -1/e$, have real part smaller or equal to -1. Supposing $z = -1 + \alpha i$ is a root of (4.8), the equation for the imaginary part yields

$$\alpha + \sin\alpha = 0$$

what implies that -1 is the only root that has -1 as real part.

It remains to show that there is a "spectral gap" $\epsilon > 0$ such that all roots $\lambda \neq -1$ of (3.20), which assumes the form

$$z + e^{-1-z} = 0, \tag{4.9}$$

satisfy $\operatorname{Re}\lambda < -1 - \epsilon$. Suppose that there is a sequence $\lambda_n \neq -1$ of roots of (4.9) such that $\operatorname{Re}\lambda_n \to -1$. Since $z \mapsto z + e^{-1-z}$ is an entire function, its zeros cannot have a finite accumulation point. So it follows that $|\operatorname{Im}\lambda_n|$ tends to infinity. But Eq. (4.9) implies that

$$|\operatorname{Im}\lambda_n| < |\lambda_n| = e^{-1-\operatorname{Re}\lambda_n} \to 1.$$

A contradiction. Therefore -1 is a dominant root of (4.9). We can derive the large time behaviour

$$\lim_{t\to\infty} te^t x(t,\varphi) = 2\left(\varphi(0) - \int_0^1 e^{\tau-1}\varphi(\tau-1)\,d\tau\right).$$

Remark 4.5. We observe that by a change of variables $x(s) = e^{-a\tau s}y(\tau s)$, the delay equation

$$\dot{x}(t) = ax(t) + bx(t-\tau), \tag{4.10}$$

with a and b real numbers, can be written on form (3.19) and have its long-time behaviour described as in Example 4.4.

4.2. A class of two-dimensional neutral FDE

We consider the FDE of the form

$$\frac{d}{dt}\big(x(t) + Nx(t-1)\big) = Bx(t) + Cx(t-1), \quad t \geqslant 0 \tag{4.11}$$

with initial condition

$$x_0 = \varphi \in \mathcal{C}\big([-1,0],\mathbb{C}^2\big), \tag{4.12}$$

where $x(t) \in \mathbb{C}^2$ and N, B and C are (2×2)-matrices with real entries. Eq. (4.11) can be written in form (1.2)

$$\frac{d}{dt}Mx_t = Lx_t, \quad t \geqslant 0$$

by setting the operators $M, L : \mathcal{C}([-1,0],\mathbb{C}^2) \to \mathbb{C}^2$ as

$$M\varphi = I\varphi(0) + N\varphi(-1), \qquad L\varphi = B\varphi(0) + C\varphi(-1). \tag{4.13}$$

It is our aim to analyze the large time behaviour of solutions of (4.11) and demonstrate the usefulness of compute spectral projections. We begin our analysis of the characteristic equations studding the right half plane were the characteristic equation has only finite many roots.

All the computations described bellow are carried out explicitly in the author's PhD thesis [6]. Given the algorithmic nature of the formula for the spectral projection, the author developed a library called *FDESpectralProj* for the computer algebra system Maple, in order to help compute characteristic equations and spectral projections.

The characteristic matrix corresponding to (4.11) is given by

$$\Delta(z) = z\Delta_0(z) - B - Ce^{-z} \tag{4.14}$$

with

$$\Delta_0(z) = I + Ne^{-z}. \tag{4.15}$$

One can obtain that the characteristic equation can be given in terms of determinants and traces of the involved matrices, as follows

$$\begin{aligned}\det\Delta(z) = {} & z^2e^{-2z}\det N + z^2e^{-z}\operatorname{tr} N + z^2 + ze^{-2z}\big(\det N + \det C - \det(N+C)\big)\\ & + ze^{-z}\big(\det N + \det B - \det(N+B) - \operatorname{tr} C\big) - z\operatorname{tr} B + e^{-2z}\det C\\ & + e^{-z}\big(\det(B+C) - \det B - \det C\big) + \det B = 0. \end{aligned} \tag{4.16}$$

We see that the characteristic equation in (4.16) depends on 9 parameters (determinants and traces of the matrices formed by sums of N, B and C), instead of the 12 coefficients of the matrices N, B and C.

From the definition of a_M in (2.6), if follows that for any $\epsilon > 0$, there are at most finitely many roots of the characteristic equation (4.16) in the right-half plane $\{z \in \mathbb{C}\colon \operatorname{Re} z > a_M + \epsilon\}$.

The following lemma presents a criterion to decide whether a real number γ is larger than a_M, given the values of $\det N$ and $\operatorname{tr} N$.

Lemma 4.6. *Suppose that* $(\det N, \operatorname{tr} N) \neq (0,0)$. *For* $\gamma \in \mathbb{R}$, *we have that* $a_M < \gamma$ *if and only if the matrix N is such that the point in the plane of coordinates* $(\det N, \operatorname{tr} N)$ *is inside the triangle of vertexes* $(-e^{2\gamma}, 0)$, $(e^{2\gamma}, 2e^{\gamma})$ *and* $(e^{2\gamma}, -2e^{\gamma})$.

Proof. First, we like to show the connection between a_M and the location of the zeros of $\det\Delta_0(z)$, where $\Delta_0(z)$ is given by (4.15). We have the following representation for $\det\Delta_0(z)$.

$$\det\Delta_0(z) = 1 + e^{-z}\operatorname{tr} N + e^{-2z}\det N. \tag{4.17}$$

Hence $\det\Delta_0(z)$ is a polynomial of degree 1 or 2 in e^{-z}. Then if z is a root of $\det\Delta_0(z)$, it follows that e^{-z} assumes at most two different values. Since the map $z \mapsto e^{-z}$ is $2\pi i$-periodic, all roots of $\det\Delta_0(z)$ are over at most two vertical lines. Let $\tilde{a}_M$ be the maximum of real parts of the roots of $\det\Delta_0(z)$. We shall show that $a_M = \tilde{a}_M$.

From (4.16), we obtain that

$$\frac{\det\Delta(z)}{z^2} = \det\Delta_0(z) + \frac{1}{z}\big(e^{-2z}c_1 + e^{-z}c_2 + c_3\big) + \frac{1}{z^2}\big(e^{-2z}d_1 + e^{-z}d_2 + d_3\big), \tag{4.18}$$

with $c_1 = \det N + \det C - \det(N+C)$, $c_2 = \det N + \det B - \det(N+B) - \operatorname{tr} C$, $c_3 = -\operatorname{tr} B$, $d_1 = \det C$, $d_2 = \det(B+C) - \det B - \det C$ and $d_3 = \det B$.

Let $\epsilon > 0$. For z such that $\operatorname{Re} z > \tilde{a}_M + \epsilon$, then e^z is uniformly bounded away from the roots of $\det\Delta_0(\cdot)$, and therefore there exists $\delta > 0$, which depends only on ϵ, such that $|\det\Delta_0(z)| > \delta$. Also for these z, we have that there exists $N > 0$ such that $|e^{-2z}c_1 + e^{-z}c_2 + c_3| < N$ and $|e^{-2z}d_1 + e^{-z}d_2 + d_3| < N$, and if we further impose $|z| > \max\{1, \frac{2N}{\delta}\}$, we obtain from (4.18) that

$$\left|\frac{\det\Delta(z)}{z^2} - \det\Delta_0(z)\right| \leqslant \left(\frac{1}{a} + \frac{1}{a^2}\right)N < \frac{2N}{a} < \delta < \big|\det\Delta_0(z)\big|.$$

From Rouché's Theorem (see for instance [15]) we obtain that $\det\Delta_0(z)$ and $\det\Delta(z)$ have the same number of zeros inside any contour contained in the region $z : \operatorname{Re} z > \tilde{a}_M + \epsilon$ and $|z| > \max\{1, \frac{2N}{\delta}\}$, that is, no zeros. Since $\det\Delta(z)$ is an entire function, its zeros must be isolated, so there are at most a finite number of zeros z such that $\operatorname{Re} z > \tilde{a}_M + \epsilon$. Therefor $a_M \leqslant \tilde{a}_M$.

Let again $\epsilon > 0$. There exists $m > 0$ such that $|\det\Delta_0(z)| > m$ for z outside circles of radius ϵ centerer in the roots of $\det\Delta_0(z)$. As before, there is $k > 0$ such that for z such that $\operatorname{Re} z > \tilde{a}_M - 1$ and $|z| > k$ and z outside circles of radius ϵ centered on the roots of $\det\Delta_0(z)$,

$$\left|\frac{\det\Delta(z)}{z^2} - \det\Delta_0(z)\right| \leqslant m < \big|\det\Delta_0(z)\big|.$$

Applying again Rouché's Theorem, we obtain that $\det\Delta(z)$ and $\det\Delta_0(z)$ have the same number of roots inside circles of radius ϵ centered on the roots of $\det\Delta_0(z)$ for $|z|$ sufficiently large. Therefore we obtain that $a_M \geqslant \tilde{a}_M$, and hence $a_M = \tilde{a}_M$.

Then $a_M < \gamma$ if and only if $\operatorname{Re} z < \gamma$ for all roots z of $\det\Delta_0(z)$, which is equivalent to $|e^{-z}| = e^{-\operatorname{Re} z} > e^{-\gamma}$. Performing technical but simple analysis on the modulus of the roots of the polynomial

$$\alpha x^2 + \beta x + 1 = 0, \quad \alpha, \beta \in \mathbb{R} \tag{4.19}$$

we obtain that, for any $\sigma > 0$, the subset of the plane of those (α, β) such the polynomial in (4.19) has σ as the minimum of the modulus of its roots, is precisely the triangle of vertexes $(-\frac{1}{\sigma^2}, 0)$, $(\frac{1}{\sigma^2}, \frac{2}{\sigma})$ and $(\frac{1}{\sigma^2}, -\frac{2}{\sigma})$. In the interior of such triangle the minimum of the modulus the roots of (4.19) is larger than σ, as one sees that, when $(\alpha, \beta) \to (0,0)$, the modulus of both roots tends to ∞. Therefore, taking $\alpha = \det N$, $\beta = \operatorname{tr} N$ and $\sigma = e^{-\gamma}$, the proof of the lemma follows. □

When N is such that $\det N = 0$ and $\operatorname{tr} N = 0$, we have that $\det\Delta_0(z) \equiv 1$! We can take advantage of this situation to conclude that the location of the roots of the characteristic equation $\det\Delta(z)$ are similar to the location of the roots of the characteristic equation of a delay equation.

Lemma 4.7. *Suppose that N is nilpotent. Then, for any vertical strip $\mathbb{C}_{\gamma_1,\gamma_2} = \{z \in \mathbb{C}\colon\ \gamma_1 < \operatorname{Re} z < \gamma_2\}$, there are at most finitely many zeros of the characteristic equation* (4.16)*. Hence, we have that $a_M = -\infty$.*

Proof. N being nilpotent implies that $\det N = 0$ and $\operatorname{tr} N = 0$. Then $\det \Delta(z)$, given in formula (4.16), can be reduced to

$$\begin{aligned}\det \Delta(z) &= z^2 + ze^{-2z}\big(\det C - \det(N+C)\big) + ze^{-z}\big(\det B - \det(N+B) - \operatorname{tr} C\big) - z \operatorname{tr} B\\ &\quad + e^{-2z} \det C + e^{-z}\big(\det(B+C) - \det B - \det C\big) + \det B.\end{aligned} \tag{4.20}$$

Define the functions $g_1(z)$ and $g_0(z)$, respectively, by

$$g_1(z) \stackrel{\text{def}}{=} e^{-2z}\big(\det C - \det(N+C)\big) + e^{-z}\big(\det B - \det(N+B) - \operatorname{tr} C\big) - \operatorname{tr} B, \tag{4.21}$$

$$g_0(z) \stackrel{\text{def}}{=} e^{-2z} \det C + e^{-z}\big(\det(B+C) - \det B - \det C\big) + \det B. \tag{4.22}$$

Combining (4.20)–(4.22), we obtain that

$$\det \Delta(z) = z^2 + g_1(z)z + g_0(z).$$

The functions g_1 and g_2 are entire and we have that there exists $C > 0$ such that $|g_1(z)| < C$ and $|g_0(z)| < C$ for all $z \in \mathbb{C}_{\gamma_1,\gamma_2}$. For $z \in \mathbb{C}_{\gamma_1,\gamma_2}$ with $|z| > C+1$, we have that

$$\begin{aligned}\big|\det \Delta(z)\big| &= \big|z^2 + g_1(z)z + g_0(z)\big| \geqslant |z|^2 - C|z| - C\\ &= \big(|z|+1\big)\left(|z| - 1 + \frac{1}{|z|+1} - C\right) > \frac{|z|+1}{|z|+1} = 1.\end{aligned}$$

Therefore, $\det \Delta(z) \neq 0$ for $z \in \mathbb{C}_{\gamma_1,\gamma_2}$ and $|z| > C+1$. Since $\det \Delta(z)$ is an entire function, its zeros are isolated and there exist at most finitely many zeros of $\det \Delta(z)$ in the vertical strip $\mathbb{C}_{\gamma_1,\gamma_2}$. □

Remark 4.8. If N is a 2×2-matrix which is nilpotent, or equivalently, such that $\det N = 0$ and $\operatorname{tr} N = 0$, then $N^2 = 0$. It is known that the spectrum of the infinitesimal generator of the neutral equation

$$\frac{d}{dt}\big(x(t) + Nx(t-1)\big) = 0$$

is closely related with the spectral properties of the difference equation

$$x(t) = -Nx(t-1), \quad t > 0. \tag{4.23}$$

The solution of the difference equation (4.23) can be build by the method of steps, *i.e.*,

$$x(t+2) = -Nx(t+1) = N^2 x(t) = 0.$$

It we start with $x_0 = \varphi$, then we have that $x(t) = 0$ for $t > 1$. Hence (4.23) has no exponential solutions. (All solutions are "small solutions"), and them the point spectrum of the infinitesimal generator is empty.

From representation (4.16), after some tedious but simple computations, one arrives to precise conditions on the parameters $\det N$, $\operatorname{tr} N$, $\det(N+B)$, $\det(N+C)$, $\det B$, $\operatorname{tr} B$, $\det(B+C)$ and $\det C$ in order to z_0 to be a zero of (4.16) of order up to 7. The condition $\operatorname{Re} z_0 > a_M$ imposes a restriction on the maximum order. We summarize these results in next theorem.

Theorem 4.9. *Let z_0 be a zero of* (4.16)*. Then z_0 is a zero of order up to eight. Furthermore, in order to have $z_0 > a_M$ it is necessary that z_0 is a zero of order at most six.*

Proof. One has that $\det \Delta(z) = 0$ is an affine equation in the parameter $\det(B+C)$. In order to z_0 to be zero of (4.16), $\det(B+C)$ must be the solution of such affine equation. Then $\frac{d}{dz} \det \Delta(z) = 0$ is an affine equation in the parameter $\det(N+B)$. So it is necessary $\det(N+B)$ to be equal to the solution of this affine equation in order to z_0 to be a zero of second order. Now $\frac{d^2}{dz^2} \det \Delta(z) = 0$ is an affine equation on the parameter $\det B$ and this parameter must be equal to the solution of that affine equation in order to z_0 to be a zero of third order. In the same way $\operatorname{tr} B$ must be the solution of an affine equation for z_0 to be a zero of fourth order.

If $z_0 \neq 3$, then $\frac{d^4}{dz^4} \det \Delta(z) = 0$ and $\frac{d^5}{dz^5} \det \Delta(z) = 0$ are respectively affine equations in the parameters $\det(N+C)$ and $\det C$. On the other hand, if $z_0 = 3$, then $\frac{d^4}{dz^4} \det \Delta(z) = 0$ and $\frac{d^5}{dz^5} \det \Delta(z) = 0$ are instead affine equations in the parameters $\det C$ and $\det(N+C)$ respectively.

So if the conditions on the parameters $\det(B+C)$, $\det(N+B)$, $\det B$, $\det(N+C)$ and $\det C$ are satisfied, then z_0 is a zero of order (at least) sixth. Under these conditions the equations $\frac{d^6}{dz^6} \det \Delta(z) = 0$ and $\frac{d^7}{dz^7} \det \Delta(z) = 0$ read

$$\frac{d^6}{dz^6}\det\Delta(z) = 8e^{-2z}\det N + 2\operatorname{tr} Ne^{-z} + 8 = 0, \tag{4.24}$$

$$\frac{d^7}{dz^7}\det\Delta(z) = -64e^{-2z}\det N - 14e^{-z}\operatorname{tr} N - 48 = 0, \tag{4.25}$$

which is linear system with unique solution $(\det N, \operatorname{tr} N) = (e^{2z_0}, -8e^{z_0})$. This shows that it is possible to (4.16) to have a zero of order up to eighth, since, under these conditions, we get that $\frac{d^8}{dz^8}\det\Delta(z) = 16$.

It remains to prove the last statement. Suppose that z_0 is a real number. Then we observe that equation $\frac{d^6}{dz^6}\det\Delta(z) = 0$ determines a straight line in the parameter space $(\det N, \operatorname{tr} N)$. This line is entirely outside the interior of the triangle described in Lemma 4.6. In fact, this line touches the border of the triangle in the vertex $(-e^{2z_0}, 0)$. So it is not possible simultaneously that $z_0 > a_M$ and $\frac{d^6}{dz^6}\det\Delta(z_0) = 0$.

Now suppose that $z_0 \notin \mathbb{R}$. Suppose at first that $\sin\operatorname{Im} z_0 = 0$. After some computations, we get that the pair $(\det N, \operatorname{tr} N)$ are over straight lines that touch the vertex $(-e^{2\operatorname{Re} z_0}, 0)$ without crossing the interior of the triangle referred in Lemma 4.6, with $\gamma = e^{\operatorname{Re} z_0}$. If $\sin\operatorname{Im} z_0 \neq 0$, the same sort of computations implies that $\det N = e^{2\operatorname{Re} z_0}$, which is in the opposite side of the referred vertex. This finishes the proof. □

Example 4.10. Actually there are examples where the maximal order of six in Theorem 4.9 is attained. We consider FDE of the form (4.11) with the matrices N, B and C given by

$$N = \begin{bmatrix} \frac{1}{2}(1+\sqrt{7}) & 2 \\ -1 & \frac{1}{2}(1-\sqrt{7}) \end{bmatrix} \approx \begin{bmatrix} 1.823 & 2 \\ -1 & -0.823 \end{bmatrix}, \tag{4.26}$$

$$B = \frac{1}{4}\begin{bmatrix} 10+\sqrt{61}-\sqrt{7} & -4 \\ 23-\frac{1}{2}\sqrt{61}\sqrt{7} & 10-\sqrt{61}+\sqrt{7} \end{bmatrix} \approx \begin{bmatrix} 3.791 & -1 \\ 3.167 & 1.209 \end{bmatrix}, \tag{4.27}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \approx \begin{bmatrix} 13.521 & 5.083 \\ -3.43 & -1.271 \end{bmatrix}, \tag{4.28}$$

where

$$\begin{aligned}
c_{11} &= \frac{49}{8} + \frac{24797\sqrt{7} + 3241\sqrt{61} + 126\sqrt{71041} - 3\sqrt{7}\sqrt{71041}\sqrt{61}}{14600}, \\
c_{12} &= \frac{24797}{3650} - \frac{3\sqrt{4333501}}{3650}, \\
c_{21} &= \frac{18467}{14600} - \frac{3241\sqrt{7}\sqrt{61}}{29200} - \frac{63\sqrt{7}\sqrt{71041}}{14600} + \frac{9\sqrt{61}\sqrt{71041}}{29200}, \\
c_{22} &= \frac{49}{8} - \frac{24797\sqrt{7} + 3241\sqrt{61} + 126\sqrt{71041} - 3\sqrt{7}\sqrt{71041}\sqrt{61}}{14600}.
\end{aligned} \tag{4.29}$$

Hence, we obtain that

$$\operatorname{tr} N = 1, \quad \det N = \frac{1}{2}, \quad \operatorname{tr} B = 5, \quad \det B = \frac{31}{4}, \quad \operatorname{tr} C = \frac{49}{4}, \quad \det C = \frac{1}{4},$$
$$\det(N+B) = 0, \quad \det(N+C) = -\frac{3}{4} \quad \text{and} \quad \det(B+C) = 0.$$

The characteristic function (4.16) becomes

$$\begin{aligned}
\det\Delta(z) &= (2z^2 + 6z + 1)\frac{e^{-2z}}{4} + (z^2 - 4z - 8)e^{-z} + z^2 - 5z + \frac{31}{4} \\
&= \left(\left(e^{-z}+1\right)^2 + 1\right)\frac{z^2}{2} + \left(3e^{-2z} - 8e^{-z} - 10\right)\frac{z}{2} + \frac{e^{-2z}}{4} - 8e^{-z} + \frac{31}{4},
\end{aligned} \tag{4.30}$$

which admits the following expansion as a Taylor's series

$$\det\Delta(z) = \frac{7}{360}z^6 - \frac{47}{2520}z^7 + \frac{5}{504}z^8 + O\left(z^9\right). \tag{4.31}$$

We have that $z_0 = 0$ is a zero of (4.30) of sixth order. Furthermore, we have that $(\det N, \operatorname{tr} N) = (\frac{1}{2}, 1)$ is inside the triangular region described in Lemma 4.6 and therefore $a_M < 0$. Indeed, in this case we have that $a_M = -\ln\sqrt{2}$. One can show that $z_0 = 0$ is actually a dominant zero.

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